# Advanced CAD/CAM
# Fall 2019

Homework 2

September 21, 2019

1. Consider the following curves:

$$\begin{aligned} f(x,y) &= -64y^4 + 128y^3 - 96x^2y^2 + 140xy^2 - 139y^2 + 96x^2y \\ &\quad -140xy + 75y - 96x^4 + 276x^3 - 313x^2 + 165x - 36 = 0, \end{aligned}$$

and

$$\mathbf{r}(t) = (x(t), y(t))^T = \sum_{i=0}^{3} \mathbf{b}_i B_{i,4}(t), \tag{1}$$

where $B_{i,4}(t)$ denotes the i-th cubic Bernstein polynomial and $\mathbf{b}_0 = (0.5, 0.5)^T$, $\mathbf{b}_1 = (0.7, 0.6)^T$, $\mathbf{b}_2 = (0.95, 0.1)^T$, $\mathbf{b}_3 = (0.55, 0.25)^T$.

(a). Compute all turning and singular points of $f(x,y)$ to the accuracy of $10^{-10}$, as well as the tangent lines at all these points.

(b). Using the results of (a) as a guide, sketch $f(x,y)$. Clearly indicate the turning and singular points on your sketch.

(c). Compute the intersections of the two curves given above to the accuracy of $10^{-10}$. In addition to giving the Cartesian coordinates of the intersection points, also include the parameter values of the points and their multiplicity.

### Note

Given an algebraic curve of $f(x,y) = 0$, we define turning and singular points as follows:

- The *x-turning points* are the points which satisfy $f(x,y) = \frac{\partial f(x,y)}{\partial y} = 0$.
- The *y-turning points* are those which satisfy $f(x,y) = \frac{\partial f(x,y)}{\partial x} = 0$.
- The *singular points* are those which satisfy $f(x,y) = \frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y} = 0$.

2. Let $\mathbf{r}(s)$ be a planar, closed and convex curve ( e.g. a circle, an ellipse, etc.) where the arc length $s$ varies in the range $[0, l]$ so that the length of the curve is $l$. Let

$$\hat{\mathbf{r}}(s) = \mathbf{r}(s) + d\mathbf{n}(s)$$

be another curve, where $d$ is a positive distance and $\mathbf{n}(s)$ is the unit normal vector of the curve $\mathbf{r}(s)$ defined by $\mathbf{n} = \mathbf{t} \times \mathbf{e}_z$. $\hat{\mathbf{r}}$ is called the interior *normal offset* at distance $d$.

**a.** Show that the total length of the curve $\hat{\mathbf{r}}(s)$ exceeds the total length of the curve $\mathbf{r}(s)$ by $2\pi d$.

**b.** Show that the area enclosed between the two curves is given by

$$A = d(l + \pi d)$$

**c.** Show that the curvatures of the two curves are related by

$$\hat{\kappa} = \frac{\kappa}{1 + d\kappa}$$

where $\kappa$ is the curvature of $\mathbf{r}(s)$ and $\hat{\kappa}$ is the curvature of the offset curve $\hat{\mathbf{r}}(s)$.

**d.** Verify your results for questions a to c for a circle of radius $R$.

3. Make plots of the B-spline basis functions of the following order $n$ (degree $= n - 1$) and knot vector T:

- n = 4, T = [0, 0, 0, 0, 1, 1, 1, 1]
- n = 4, T = [0, 0, 0, 0, 2, 4, 4, 4, 4]
- n = 4, T = [0, 0, 0, 0, 3, 3, 6, 6, 6, 6]
- n = 4, T = [0, 0, 0, 0, 2, 2, 2, 6, 6, 6, 6]

- n = 4,  T = [0, 0, 0, 0, 1, 2, 3, 4, 6, 7, 7, 7, 7]
- n = 3,  T = [0, 0, 0, 3, 6, 9, 9, 9]
- n = 2,  T = [0, 0, 2, 4, 4]

You may use *MATLAB* or Python for your plots.

4. Given a list of Cartesian points in 3-D space which represent a non-periodic curve, construct a cubic Bézier curve using least squares approximation of the points. Also, construct a cubic B-spline curve with non-uniform knots using least squares approximation of these points. Include a simple visualization of the results.