Engineering Analysis

Final Exam

June 13, 2017

1. (5 pts) Consider a continuous function $f(x)$. Derive the first-order derivative formulae at $x = x_0$ using the forward and central difference methods, respectively. You need to provide all the intermediate steps to get a full score.

2. (8 pts) Show that the fourth-order Runge-Kutta method,

$$\begin{aligned} k_1 &= hf(t_i, \omega_i), \\ k_2 &= hf(t_i + h/2, \omega_i + k_1/2), \\ k_3 &= hf(t_i + h/2, \omega + k_2/2), \\ k_4 &= hf(t_i + h, \omega_i + k_3), \\ \omega_{i+1} &= \omega_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned}$$

when applied to the differential equation $y' = \lambda y$, can be written in the form

$$\omega_{i+1} = (1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4)\omega_i.$$

3. (15 pts) Suppose that we have an ODE $y' = -5y$ with the initial condition $y(0) = 1$. Using the stepsize $h = 0.5$, we solve the ODE numerically. Then answer the following questions.

   (a) (3 pts) Describe the trade-offs involved in using an explicit versus implicit method for integrating an initial value problem. Under what circumstances does one or the other have a clear advantage?

   (b) (6 pts) Determine whether Euler's method is stable for this ODE using the stepsize $h = 0.5$.

   (c) (6 pts) If we use the implicit Euler's method, then the method is stable using $h = 0.5$? Why?

4. (15 pts) Consider a second order differential equation $y'' + 3y' + 2y = 0$.

   (a) (5 pts) Represent the given ODE as a first-order system of ODEs.

   (b) (10 pts) Using the second order Taylor's method, solve the system of ODEs with the initial conditions $y(0) = 1$, $y'(0) = 2$ and the stepsize $h = 0.01$. Carry out one iteration of the computation.

5. (7 pts) In Golden Section search method, estimate the number of iterations needed to achieve a precision of 0.001 given the initial interval of $[0, 1]$.

6. (10 pts) Consider the second order differential equations $y'' = 6t$ with the boundary conditions $y(0) = 0$ and $y(1) = 1$. Suppose that the finite difference approximation for the second-order derivative is

$$y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

Also, the interval is divided into two equal subintervals. Then, compute the value of y at $t = 0.5$.

7. (20 pts) Consider the linear system

$$\begin{aligned} x_1 + \frac{1}{2}x_2 &= \frac{3}{2} \\ \frac{1}{2}x_1 + \frac{1}{3}x_2 &= \frac{5}{6} \end{aligned}$$

and answer the following questions using 2-digit chopping arithmetic.

(a) (5 pts) Suppose that the vector $\hat{\mathbf{x}} = (\hat{x_1}, \hat{x_2}) = (1.4, 0.35)$ is an approximate solution. Compute the residual error vector.

(b) (5 pts) Compute the exact solution using the Gaussian elimination.

(c) (10 pts) Using the Jacobi iterative method, compute the solution using $\hat{\mathbf{x}}$ as the starting vector. Does the Jacobi iterative method improve the accuracy?

8. (5 pts) Derive Simpson's rule with error term by using

$$\int_{x_0}^{x_2} f(x)dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(k)}(\xi).$$

Find a_0, a_1 and a_2 from the fact that Simpson's rule is exact for $f(x) = x^n$ when $n = 1, 2$, and 3. Then find k by applying the integration formula with $f(x) = x^4$.

9. (15 pts) Perform the singular value decomposition on the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}.$$