Engineering Analysis
Spring 2017
Boundary Value Problems and Partial Differential Equations *

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1 Two Point Boundary Value Problems

When ordinary differential equations are required to satisfy boundary conditions at more than one value of the independent variable, the resulting problem is called *a two point boundary value problem*.

We will consider second-order equations with two boundary values in this lecture.

The two-point boundary-value problems considered here involve a second-order differential equation of the form

$$y'' = f(x, y, y'), a \le x \le b,$$

together with the boundary conditions

$$y(a) = \alpha, \text{ and } y(b) = \beta.$$

1.1 The Linear Shooting Method

Theorem 1.1 *Suppose the function f in the boundary-value problem*

$$y'' = f(x, y, y'), a \le x \le b, y(a) = \alpha, y(b) = \beta,$$

is continuous on the set

$$D = \{(x, y, y')|a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\},$$

and that the partial derivatives f_y and $f_{y'}$ are also continuous on D. If

- *$f_y(x, y, y') > 0$, for all $(x, y, y') \in D$, and*
- *a constant M exists, with*

$$|f_{y'}(x, y, y'| \le M, \text{ for all } (x, y, y') \in D,$$

*This section has been produced based on the following books, *Numerical Analysis, 8th Edition, by R. L. Burden and J. D. Faires* , and *Numerical Recipes in C, The Art of Scientific Computing, Second Edition, by W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery.*

then the boundary-value problem has a unique solution.

When $f(x, y, y')$ has the form

$$f(x, y, y') = p(x)y' + q(x)y + r(x),$$

the differential equation

$$y'' = f(x, y, y')$$

is *linear*.

Corollary

If the linear boundary-value problem

$$y'' = p(x)y' + q(x)y + r(x), a \le x \le b, y(a) = \alpha, y(b) = \beta,$$

satisfies

- $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$,
- $q(x) > 0$ on $[a, b]$,

then the problem has a unique solution.

To approximate the unique solution guaranteed by the satisfaction of the hypotheses of the corollary, first consider the initial-value problems

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, \quad y'(a) = 0, \tag{1}$$

and

$$y'' = p(x)y' + q(x)y, \quad a \le x \le b, \quad y(a) = 0, \quad y'(a) = 1, \tag{2}$$

It can be proved that both problems have a unique solution.

Let $y_1(x)$ denote the solution to Eq. (1), $y_2(x)$ denote the solution to Eq. (2), and let

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x). \tag{3}$$

Then

$$y' = y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2'(x)$$

and

$$y'' = y_1''(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2''(x)$$

So, we have $y'' = p(x)y'(x) + q(x)y(x) + r(x)$ using Eqs. (1) and (2). Also, $y(a) = \alpha$ and $y(b) = \beta$. Here, $y(x)$ is the unique solution to the linear boundary-value problem, provided that $y_2(b) \ne 0$.

The Shooting method for linear equations is based on the replacement of the linear boundary-value problem by the two initial-value problems (1) and (2).

Graphically, the method has the appearance as shown in Fig. 1.

You may use the fourth order Runge-Kutta method to find the approximations to $y_1(x)$ and $y_2(x)$.

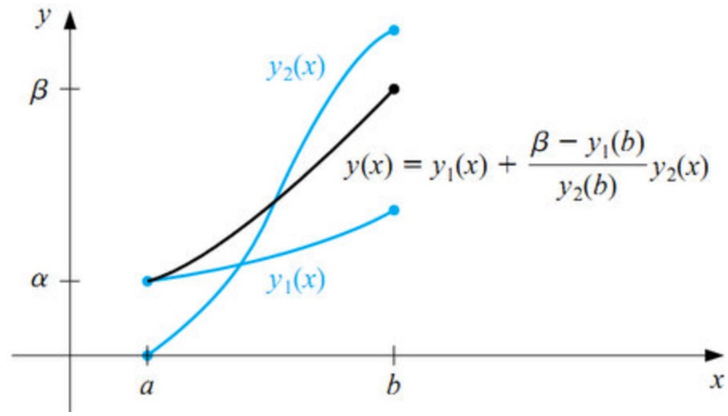


Figure 1: Illustration of the linear shooting method

1.2 The Shooting Method for Nonlinear Problems

The shooting technique for the nonlinear second-order boundary-value problem

$$y'' = f(x, y, y'), a \leq x \leq b, y(a) = \alpha, y(b) = \beta, \tag{4}$$

is similar to the linear technique, except that the solution to a nonlinear problem cannot be expressed as a linear combination of the solutions to two initial-value problems. Instead, we approximate the solution to the boundary-value problem by using the solutions to a sequence of initial-value problems involving a parameter t. These problems have the form

$$y'' = f(x, y, y'), a \leq x \leq b, y(a) = \alpha, y(b) = t. \tag{5}$$

We do this by choosing the parameters $t = t_k$ in a manner to ensure that

$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta,$$

where $y(x, t_k)$ denotes the solution to the initial-value problem (5) with $t = t_k$, and $y(x)$ denotes the solution to the boundary-value problem (4).

This technique is called a *shooting* method. We start with a parameter t_0 that determines the initial elevation at which the object is *fired* from the point (a, α) and along the curve described by the solution to the initial-value problem:

$$y'' = f(x, y, y'), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = t_0.$$

If $y(b, t_0)$ is not sufficiently close to β, we correct our approximation by choosing elevations t_1, t_2 and so on, until $y(b, t_k)$ is sufficiently close to hitting β as shown in Fig. 2.

Suppose $y(x, t)$ denotes the solution to the initial-value problem (5), we next determine t with

$$y(b, t) - \beta = 0. \tag{6}$$

This is a nonlinear equation, so a number of methods are available. However, the powerful Newton's method is employed in general.

Suppose we rewrite the initial-value problem (5), emphasizing that the solution depends on both x and the parameter t:

$$y'' = f(x, y(x, t), y'(x, y)), \quad a \leq x \leq b, \quad y(a, t) = \alpha, \quad y'(a, t) = t. \tag{7}$$

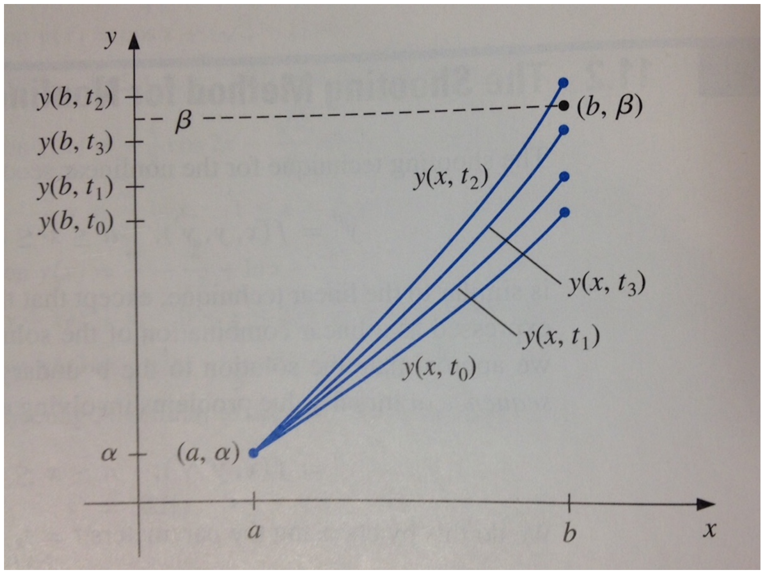


Figure 2: Illustration of the nonlinear shooting method

Since we need to determine $(dy/dt)(b,t)$ when $t = t_{k-1}$, we first take the partial derivative of Eq. (7) with respect to t, yielding

$$\begin{aligned} \frac{\partial y''}{\partial t}(x,t) &= \frac{\partial f}{\partial t}(x, y(x,t), y'(x,t)) \\ &= \frac{\partial f}{\partial x}(x, y(x,t), y'(x,t))\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\frac{\partial y}{\partial t}(x,t) \\ &+ \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\frac{\partial y'}{\partial t}(x,t). \end{aligned} \tag{8}$$

Since x and t are independent, $\partial x/\partial t = 0$ and

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\frac{\partial y}{\partial t}(x,t) + \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\frac{\partial y'}{\partial t}(x,t), \tag{9}$$

for $a \leq t \leq b$. The initial conditions give $\frac{\partial y}{\partial t}(a,t) = 0$ and $\frac{\partial y'}{\partial t}(a,t) = 1$.

If we simplify the notation using $z(x,t)$ to denote $(\partial y/\partial t)(x,t)$ and assume that the order of differentiation of x and t can be reversed, Eq. (9) with the initial conditions becomes the initial-value problem

$$z''(x,t) = \frac{\partial f}{\partial y}(x,y,y')z(x,t) + \frac{\partial f}{\partial y'}(x,y,y')z'(x,t), \tag{10}$$

$$a \leq x \leq b, \quad z(a,t) = 0, \quad z'(a,t) = 1.$$

Newton's method therefore requires that two initial-value problems be solved for each iteration, (7) and (10). Then,

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{z(b, t_{k-1})}. \tag{11}$$

Note that none of these initial value problems is solved exactly. The linear and nonlinear Shooting methods for boundary-value problems can present problems of instability.

1.3 Finite-Difference Methods for Linear Problems

The instability problem of the Shooting methods can be avoided by the finite-difference methods.

Methods involving finite differences for solving boundary-value problems replace each of the derivatives in the differential equation with an appropriate difference-quotient approximation.

Consider the general nonlinear boundary-value problem

$$y'' = f(x, y, y'), \ \ a \le x \le b, \ \ y(a) = \alpha, \ \ y(b) = \beta.$$

We divide $[a, b]$ into $(N + 1)$ equal subintervals whose endpoints are at $x_i = a + ih$, for $i = 0, 1, \cdots, N+1$. Assuming that the exact solution has a bounded fourth derivative allows us to replace $y'\prime(x_i)$ and $y'(x_i)$ in each of the equations

$$y''(x_i) = f(x_i, y(x_i), y'(x_i))$$

by the appropriate centered-difference formula. This gives, for each $i = 1, 2, \cdots, N$,

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = f\left(x_i, y(x_i), \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y'''(\eta)\right) + \frac{h^2}{12}y^{(4)}(\xi_i),$$

for some ξ_i and η_i in the interval (x_{i-1}, x_{x+1}).

The difference method results when the error terms are deleted and the boundary conditions are employed:

$$\omega_0 = \alpha, \ \ \omega_{N+1} = \beta,$$

and

$$-\frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{h^2} + f\left(x_i, \omega_i, \frac{\omega_{i+1} - \omega_{i-1}}{2h}\right) = 0,$$

for each $i = 1, 2, \cdots, N$. From this method, we obtain $N \times N$ nonlinear system, which is solved by Newton's method. A sequence of iterates $\{(\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_N^{(k)})^t\}$ is generated that converges to the solution, provided that the initial approximation $(\omega_1^{(k)}, \omega_2^{(k)}, \cdots, \omega_N^{(k)})^t$ is sufficiently close to the solution, and that the Jacobian matrix for the system is nonsingular. The Jacobian matrix

$$J(\omega_1, \cdots, \omega_N)_{ij} = \begin{cases} -1 + \frac{h}{2} f_{y'}\left(x_i, \omega_i, \frac{\omega_{i+1}-\omega_{i-1}}{2h}\right), & i = j-1, j = 2, \cdots, N, \\ 2 + h^2 f_y\left(x_i, \omega_i, \frac{\omega_{i+1}-\omega_{i-1}}{2h}\right), & i = j, j = 1, \cdots, N, \\ -1 - \frac{h}{2} f_{y'}\left(x_i, \omega_i, \frac{\omega_{i+1}-\omega_{i-1}}{2h}\right), & i = j+1, j = 1, \cdots, N-1, \end{cases} \quad (12)$$

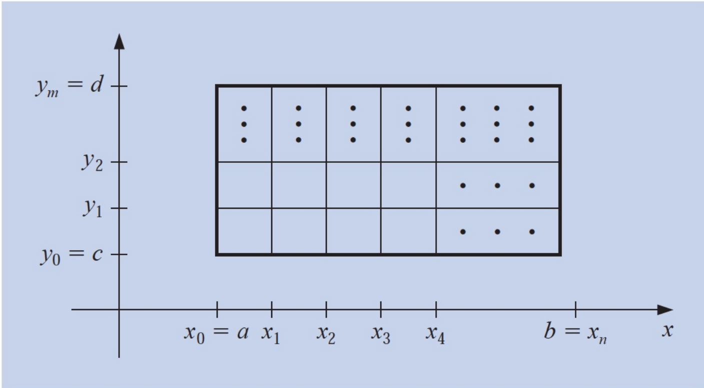
where $\omega_0 = \alpha$ and $\omega_{N+1} = \beta$.

Newton's method for nonlinear systems requires that at each iteration the $N \times N$ linear system

$$\begin{aligned} J(\omega_1, \cdots, \omega_N)(v_1, \cdots, v_n)^t \ &= \ -\left(2\omega_1 - \omega_2 - \alpha + h^2 f\left(x_1, \omega_1, \frac{\omega_2 - \alpha}{2h}\right),\right. \\ & \quad -\omega_1 + 2\omega_2 - \omega_3 + h^2 f\left(x_2, \omega_2, \frac{\omega_3 - \omega_1}{2h}\right), \cdots, \\ & \quad \left. -\omega_{N-1} + 2\omega_N + h^2 f\left(x_N, \omega_N, \frac{\beta - \omega_{N-1}}{2h}\right) - \beta\right)^t, \end{aligned}$$

be solved for $v_1, v_2, \cdots, v_N$, since

$$\omega_i^{(k)} = \omega_i^{(k-1)} + v_i, \ \ \text{for each} \ \ i = 1, 2, \cdots, N.$$



2 Partial Differential Equations

The topic of solving a partial differential equation (PDE) numerically is a vast subject. PDEs are formulated as governing equations in various fields such as fluids, electromagnetic fields, the human body, etc.

Partial differential equations (PDEs) are classified into the three categories, hyperbolic, parabolic, and elliptic, on the basis of their characteristics. In this course, we will focus on a numerical method for solving an elliptic partial differential equation.

The *elliptic* partial differential equation considered here is the Poisson equation,

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 v}{\partial y^2}(x,y) = f(x,y) \tag{13}$$

on $R = \{(x,y)|a < x < b, c < y < d\}$, with

$$u(x,y) = g(x,y) \text{ for } (x,y) \in S,$$

where S denotes the boundary of R. If f and g are continuous on their domains, then there is a unique solution to this equation.

The finite difference method is employed to solve this problem. The first step is to choose integers n and m and define step sizes $h = (b-a)/n$ and $k = (d-c)/m$. Next, we partition the interval $[a,b]$ into n equal parts of width h and the interval $[c,d]$ into m equal parts of width k.

Place a grid on the rectangle R by drawing vertical and horizontal lines through the points with coordinates (x_i, y_i), where

$$x_i = a + ih \text{ for each } i = 0, 1, \cdots, n,$$

and

$$y_i = c + jk \text{ for each } j = 0, 1, \cdots, m.$$

For each mesh point, (x_i, y_j), for $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, we use the Taylor series in the variable x about x_i to generate the centered-difference formula

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j), \tag{14}$$

where $\xi_i \in (x_{i-1}, x_{i+1})$. Similarity, we apply the Taylor series in the variable y about y_j to produce

$$\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j), \tag{15}$$

where $\eta_j \in (y_{j-1}, x_{j+1})$. Then the Poisson equation can be given at the points (x_i, y_j) as

$$\begin{aligned} & \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} \\ = \quad & f(x_i, y_j) + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j), \end{aligned}$$

for $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$. The boundary conditions are given by

$$u(x_0, y_j) = g(x_0, y_j) \text{ and } u(x_n, y_j) = g(x_n, y_j), \text{ for each } j = 0, 1, \cdots, m;$$

$$u(x_i, y_0) = g(x_i, y_0) \text{ and } u(x_i, y_m) = g(x_i, y_m), \text{ for each } i = 0, 1, \cdots, n-1;$$

These difference-equation form results in the *Finite Difference Method* as follows:

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right]\omega_{ij} - (\omega_{i+1,j} + \omega_{i-1,j}) - \left(\frac{h}{k}\right)^2(\omega_{i,j+1} + \omega_{i,j-1}) = -h^2 f(x_i, y_j), \qquad (16)$$

for each $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, and

$$\begin{aligned} \omega_{0j} &= g(x_0, y_j) \text{ and } \omega_{nj} = g(x_n, y_j), \text{ for each } j = 0, 1, \cdots, m; \qquad (17) \\ \omega_{i0} &= g(x_i, y_0) \text{ and } \omega_{im} = g(x_i, y_m), \text{ for each } i = 0, 1, \cdots, n-1; \end{aligned}$$

Here, ω_{ij} approximates $u(x_i, y_j)$.