Surfaces and Advanced Topics

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Contents

Section 2

Differential geometry of surfaces

2.1 Introduction

Differential geometry is concerned with those properties of surfaces which depend on their behavior in a neighborhood of a point [5]. For this purpose, we need a mathematical definition of a surface, which is presented in the following section.

2.2 Definition of surfaces

Roughly speaking, a *regular* surface in $\mathbf{R}^3$ is obtained by taking pieces of a plane, deforming, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure [5].

The correct mathematical definition of a surface is given by the following:

Definition 2.2.1 *A subset* $\mathbf{S} \subset \mathbf{R}^3$ *is a regular surface if, for each* $\mathbf{p} \in \mathbf{S}$*, there exists a neighborhood* $\mathbf{V}$ *in* $\mathbf{R}^3$ *and a map* $\mathbf{x} : \mathbf{U} \rightarrow \mathbf{V} \cap \mathbf{S}$ *of an open set* $\mathbf{U} \subset \mathbf{R}^2$ *onto* $\mathbf{V} \cap \mathbf{S} \subset \mathbf{R}^3$ *such that*

1. $\mathbf{x}$ *is differentiable.*
2. $\mathbf{x}$ *is a homeomorphism.*
3. *For each* $\mathbf{q} \in \mathbf{U}$*, the differential* $d\mathbf{x}_q : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ *is one-to-one.*

Equivalently, Condition 3 means that the vector cross product $\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$. Figure 2.2 shows some cases where the regularity condition is violated.

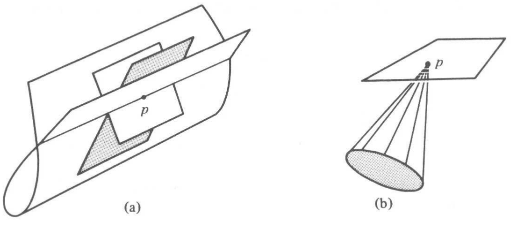


Figure 2.1: Situations to be avoided in the defintion of a regular surface

- *Implicit surfaces* $F(x, y, z) = 0$
  Example: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Ellipsoid, see Figure 2.2.

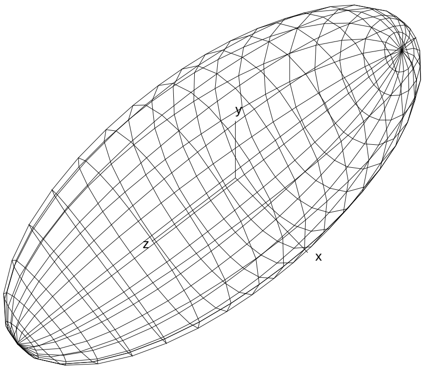


Figure 2.2: Ellipsoid.

- *Explicit surfaces* If the implicit equation $F(x, y, z) = 0$ can be solved for one of the variables as a function of the other two, we obtain an explicit surface, as shown in Figure 2.3.
  Example: $z = \frac{1}{2}(\alpha x^2 + \beta y^2)$

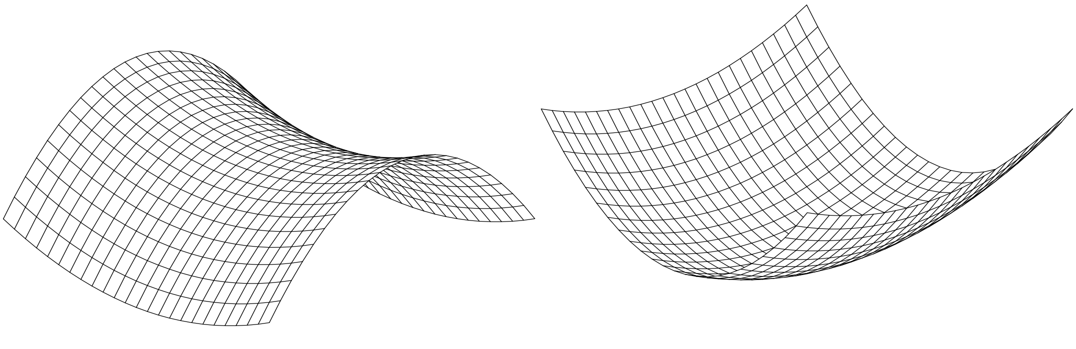


Figure 2.3: Explicit quadratic surfaces $z = \frac{1}{2}(\alpha x^2 + \beta y^2)$. (a) Left: Hyperbolic paraboloid ($\alpha = -3$, $\beta = 1$). (b) Right: Elliptic paraboloid ($\alpha = 1$, $\beta = 3$).

- *Parametric surfaces* $x = x(u, v),\ y = y(u, v),\ z = z(u, v)$
  Here functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives of the r^{th} order, and the parameters u and v are restricted to some intervals (i.e., $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$) leading to parametric surface patches. This rectangular domain D of u, v is called *parametric space* and it is frequently the unit square, see Figure 2.4.

  The rigorous mathematical definition of a parametrized surface is given as follows:

  Definition 2.2.2 *A parametrized surface* $\mathbf{x} : \mathbf{U} \subset \mathbf{R}^2 \to \mathbf{R}^3$ *is a differentiable map* $\mathbf{x}$ *from an open set* $\mathbf{U} \subset \mathbf{R}^2$ *into* $\mathbf{R}^3$. *The set* $\mathbf{x}(\mathbf{U}) \subset \mathbf{R}^3$ *is called the trace of* $\mathbf{x}$. $\mathbf{x}$ *is regular if the differential* $d\mathbf{x}_q : \mathbf{R}^2 \to \mathbf{R}^3$ *is one-to-one for all* $\mathbf{q} \in \mathbf{U}$. *A point* $\mathbf{q} \in \mathbf{U}$ *where* $d\mathbf{x}_q$ *is not one-to-one is called a singular point of* $\mathbf{x}$.

  If derivatives of the surface are continuous up to the r^{th} order, the surface is said to be of class r, denoted C^r.

In vector notation:

$$\begin{aligned}&\mathbf{r} = \mathbf{r}(u, v)\\ &\textit{where} \quad \mathbf{r} = (x, y, z), \quad \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))\end{aligned}$$

Example:

$$\mathbf{r} = (u + v, u - v, u^2 + v^2)$$

$$\left.\begin{aligned}x &= u + v\\ y &= u - v\\ z &= u^2 + v^2\end{aligned}\right\} \Rightarrow \text{eliminate } u, v \Rightarrow z = \frac{1}{2}(x^2 + y^2) \ \text{ paraboloid}$$

2.3 Curves on a surface

Let $\mathbf{r} = \mathbf{r}(u, v)$ be the equation of a surface, defined on a domain D (i.e., $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$). Let $\beta(t) = (u(t), v(t))$ be a curve in the parameter plane. Then $\mathbf{r} = \mathbf{r}(u(t), v(t))$ is a curve lying on the surface, see Figure 2.4. A tangent vector of curve $\beta(t)$ is given by $\dot{\beta}(t) = (\dot{u}(t), \dot{v}(t))$. Subsequently, a tangent vector of a curve on a surface is given by:

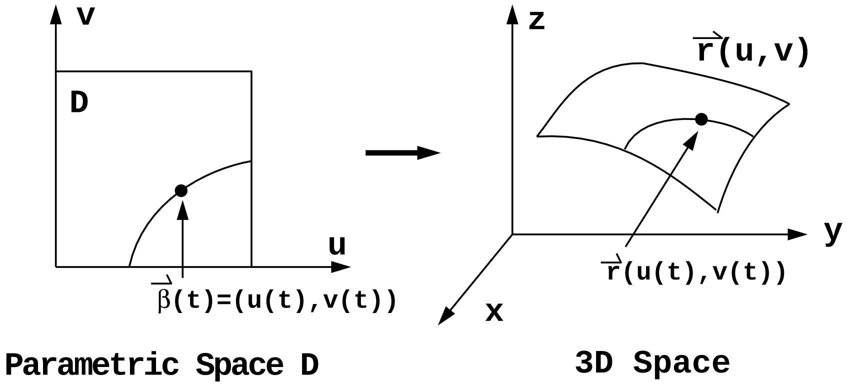


Figure 2.4: The mapping of a curve in 2D parametric space onto a 3D biparametric surface.

$$\frac{d\mathbf{r}(u(t), v(t))}{dt} \tag{2.1}$$

By using the chain rule:

$$\frac{d\mathbf{r}(u(t), v(t))}{dt} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{dt} = \mathbf{r}_u \dot{u}(t) + \mathbf{r}_v \dot{v}(t). \tag{2.2}$$

2.4 First fundamental form (arc length)

Consider a curve on a surface $\mathbf{r} = \mathbf{r}(u(t), v(t))$. The arc length of the curve on a surface is given by

$$\begin{aligned}ds &= |\frac{d\mathbf{r}}{dt}|dt = |\mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}|dt\\ &= \sqrt{(\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}) \cdot (\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v})}dt\\ &= \sqrt{(\mathbf{r}_u \cdot \mathbf{r}_u)du^2 + 2\mathbf{r}_u\mathbf{r}_v du dv + (\mathbf{r}_v \cdot \mathbf{r}_v)dv^2}\\ &= \sqrt{E du^2 + 2F du dv + G dv^2} \qquad (2.3)\end{aligned}$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v \tag{2.4}$$

The first fundamental form is defined as

$$\begin{aligned} I &= d\mathbf{r} \cdot d\mathbf{r} = (\mathbf{r}_u du + \mathbf{r}_v dv) \cdot (\mathbf{r}_u du + \mathbf{r}_v dv) \\ &= Edu^2 + 2Fdudv + Gdv^2 \end{aligned} \tag{2.5}$$

E, F, G are called the first fundamental form coefficients Note that $E = \mathbf{r}_u \cdot \mathbf{r}_u > 0$ and $G = \mathbf{r}_v \cdot \mathbf{r}_v > 0$ if $\mathbf{r}_u \neq 0$ and $\mathbf{r}_v \neq 0$. The first fundamental form I is positive definite. That is $I \geq 0$ and $I = 0$ if and only if $du = 0$ and $dv = 0$ since

$$I = \frac{1}{E}(E\ du + F\ dv)^2 + \frac{EG - F^2}{E} dv^2 \text{ and } EG - F^2 = |r_u \times r_v|^2 > 0.$$

I depends only on the surface and not on the parametrization.
The area of the surface can be derived as follows:

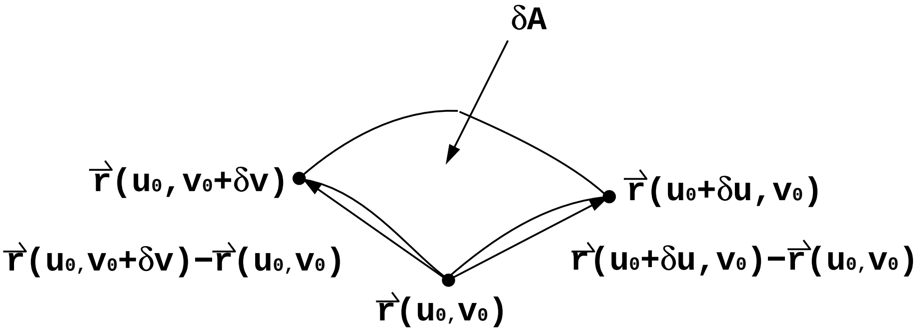


Figure 2.5: The area of an infinitessimal surface patch.

$$\mathbf{r}(u_0, v_0 + \delta v) - \mathbf{r}(u_0, v_0) \simeq \frac{\partial \mathbf{r}}{\partial v}\delta v$$
$$\mathbf{r}(u_0 + \delta u, v_0) - \mathbf{r}(u_0, v_0) \simeq \frac{\partial \mathbf{r}}{\partial u}\delta u$$

$$\delta A = |\mathbf{r}_u \delta u \times \mathbf{r}_v \delta v| = |\mathbf{r}_u \times \mathbf{r}_v| \delta u \delta v$$

$$|\mathbf{r}_u \times \mathbf{r}_v|^2 = (\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v)$$

Using the vector identity $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$, we get

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v|^2 &= (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 && (2.6) \\ &= EG - F^2 && (2.7) \end{aligned}$$

$$\delta A = \sqrt{EG - F^2}\ \delta u \delta v, \qquad A = \int\int \sqrt{EG - F^2}\ dudv \tag{2.8}$$

Example: For the hyperbolic paraboloid $\mathbf{r}(u, v) = (u, v, u^2 - v^2)$, let us derive an expression for the area of a region of its surface corresponding to a the circle $u^2 + v^2 \leq 1$ in the parametric domain D.

We begin by forming expressions for the derivatives of the position vector $\mathbf{r}$ and the first fundamental form coeffients.

$$\begin{aligned}
\mathbf{r}_u &= (1, 0, 2u) \\
\mathbf{r}_v &= (0, 1, -2v) \\
E &= \mathbf{r}_u \cdot \mathbf{r}_u = 1 + 4u^2 \\
F &= \mathbf{r}_u \cdot \mathbf{r}_v = -4uv \\
G &= \mathbf{r}_v \cdot \mathbf{r}_v = 1 + 4v^2
\end{aligned}$$

Using Equation (2.8), we find

$$\begin{aligned}
EG - F^2 &= (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2 > 0 \\
A &= \int\int_D \sqrt{1 + 4u^2 + 4v^2} du dv
\end{aligned}$$

To compute the area, we need to evaluate the double integral over the unit disk $u^2 + v^2 \leq 1$ in the parametric domain D;

$$A = \int\int_{u^2+v^2 \leq 1} \sqrt{1 + 4u^2 + 4v^2}\, du\, dv.$$

To perform the integration, let us change variables.

$$\begin{aligned}
u &= r\cos(\theta),\ v = r\sin(\theta), \text{and } du\, dv = r\, dr\, d\theta \\
A &= \int\int_{r \leq 1} \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\
&= \frac{\pi}{6}(5\sqrt{5} - 1)
\end{aligned}$$

2.5 Tangent plane

Condition 3 in the definition of a regular surface $\mathbf{S}$ guarantees that for every $\mathbf{p} \in \mathbf{S}$ the set of tangent vectors to the parametrized curves of $\mathbf{S}$, passing through $\mathbf{p}$, constitutes a plane [5]. By a tangent vector to $\mathbf{S}$, at a point $\mathbf{p} \in \mathbf{S}$, we mean the tangent vector $\alpha'(0)$ of a differentialble parametrized curve $\alpha : (-\epsilon, \epsilon) \rightarrow \mathbf{S}$ with $\alpha(0) \rightarrow \mathbf{p}$. Therefore, a tangent plane at a point $\mathbf{r}(u_o, v_o)$ is the union of tangent vectors of all curves on the surface passing through $\mathbf{r}(u_o, v_o)$, as shown in Figure 2.7. Since the tangent vector of a curve on a parametric surface is given by $\frac{d\mathbf{r}}{dt} = \mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}$, the tangent plane lies on the plane of the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. The equation of the tangent plane is

$$\mathbf{T}_p(u, v) = \mathbf{r}(u, v) + \lambda \mathbf{r}_u(u, v) + \mu \mathbf{r}_v(u, v) \tag{2.9}$$

where λ and μ are real variables parameterizing the plane.

The tangent plane also allows us to speak of the angle of two intersecting surfaces at a point of intersection [5]. Namely, the angle of two intersecting surfaces at an intersection point is the angle of their tangent planes (or their normal lines) at the point.

2.6 Normal vector

The surface normal is the vector at point $\mathbf{r}(u_o, v_o)$ perpendicular to the tangent plane, see Figure 2.8. And therefore

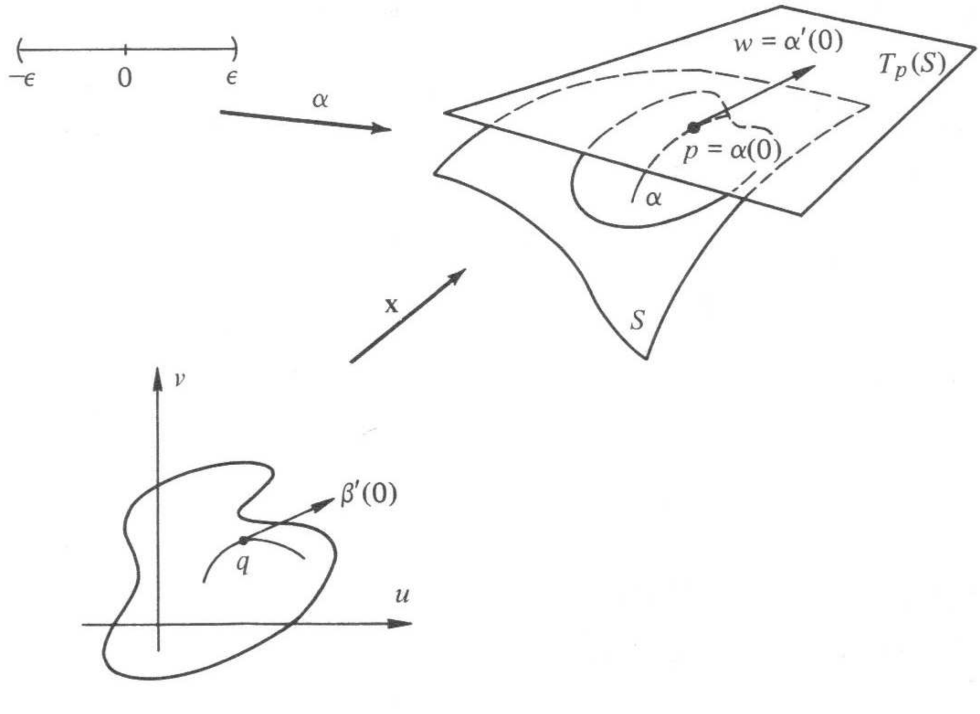


Figure 2.6: The tangent plane of a surface at p

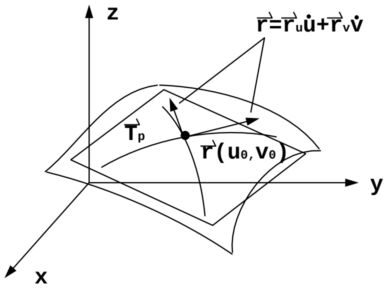


Figure 2.7: The tangent plane at a point on a surface.

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \tag{2.10}$$

Note that $\mathbf{r}_u$ and $\mathbf{r}_v$ are not necessarily perpendicular.

A *regular* (ordinary) point $\mathbf{P}$ on the surface is defined as one for which $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$. A point where $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{0}$ is called a *singular* point. The condition $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ requires that at that point $\mathbf{P}$ the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ do not vanish and have different directions.

Example: Elliptic Paraboloid $\mathbf{r}(u, v) = (u + v, u - v, u^2 + v^2)$

$$\begin{aligned}
\mathbf{r}_u &= (1, 1, 2u) \\
\mathbf{r}_v &= (1, -1, 2v)
\end{aligned}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 1 & 2u \\ 1 & -1 & 2v \end{vmatrix}$$

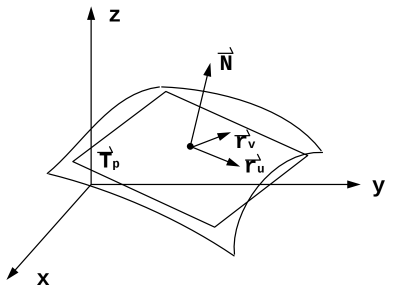


Figure 2.8: The normal to the point on a surface.

$$
\begin{aligned}
&= 2(u+v)\mathbf{e}_x + 2(u-v)\mathbf{e}_y - 2\mathbf{e}_z \neq \mathbf{0} \\
|\mathbf{r}_u \times \mathbf{r}_v| &= 2\sqrt{(u+v)^2 + (u-v)^2 + 1} \\
&= 2\sqrt{2u^2 + 2v^2 + 1} > 0 \Rightarrow \text{Regular !} \\
\mathbf{N} &= \frac{(2(u+v), 2(u-v), -2)}{2\sqrt{2u^2 + 2v^2 + 1}} \\
&= \frac{(u+v, u-v, -1)}{\sqrt{2u^2 + 2v^2 + 1}} \\
&\text{at } (u,v) = (0,0), \mathbf{N} = (0,0,-1)
\end{aligned}
$$

Example: Circular Cone $\mathbf{r}(u,v) = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha)$, see Figure 2.9

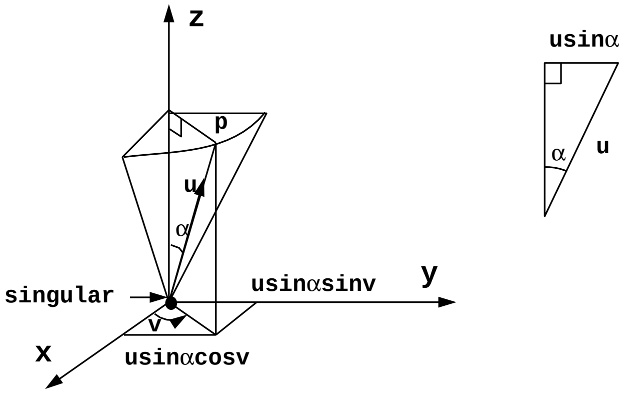


Figure 2.9: Circular cone.

$$
\begin{aligned}
\mathbf{r}_u &= (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha) \\
\mathbf{r}_v &= (-u\sin\alpha sinv, u\sin\alpha sinv, 0)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \sin\alpha\cos v & \sin\alpha\sin v & \cos\alpha \\ -u\sin\alpha\sin v & u\sin\alpha\cos v & 0 \end{vmatrix} \\
&= -u\sin\alpha\cos\alpha\cos v\mathbf{e}_x - u\sin\alpha\cos\alpha\sin v\mathbf{e}_y + u\sin^2\alpha\mathbf{e}_z
\end{aligned}
$$

At the origin $\mathbf{n} = 0$,

$$\mathbf{r}_u \times \mathbf{r}_v \;=\; \mathbf{0}$$

Therefore, the apex of the cone is a singular point.

2.7 The Geometry of the Gauss Map

For a curve, by considering the rate of change of the tangent line of a curve $\mathbf{C}$, we define an important geometric entity, i.e. the curvature of $\mathbf{C}$. The same idea can be extended to a regular surface to define the *curvature* of the surface. Conceptually, we can measure how rapidly a surface $\mathbf{S}$ pulls away from the tangent plane $\mathbf{T}_p(\mathbf{S})$ in a neighborhood of a point $\mathbf{p} \in \mathbf{S}$. This is equivalent to measuring the rate of change at $\mathbf{p}$ of a unit normal vector field $\mathbf{N}$ on a neighborhood of $\mathbf{p}$.

2.7.1 Definition of the Gauss Map and Its Fundamental Properties

Given a parametrization $\mathbf{r} : \mathbf{U} \subset \mathbf{R}^2 \rightarrow \mathbf{S}$ of a regular surface $\mathbf{S}$ at a point $\mathbf{p} \in \mathbf{S}$, we have a differentiable map $\mathbf{N} : \mathbf{r}(U) \rightarrow \mathbf{R}^3$

$$\mathbf{N}(\mathbf{q}) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}(q), \tag{2.11}$$

a unit normal vector on $\mathbf{S}$ at $\mathbf{q}$. We shall say that a regular surface is *orientable* if it admits a differentiable field of unit normal vectors defined on the whole surface. The choice of such a field $\mathbf{N}$ is called an *orientation* of $\mathbf{S}$.

Definition 2.7.1 *Let* $\mathbf{S} \subset \mathbf{R}^3$ *be a surface with an orientation* $\mathbf{N}$. *The map* $\mathbf{N} : \mathbf{S} \rightarrow \mathbf{R}^3$ *takes its values in the unit sphere*

$$\mathbf{S}^2 = \left\{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\right\}. \tag{2.12}$$

The map $\mathbf{N} : \mathbf{S} \rightarrow \mathbf{S}^2$, *thus defined, is called the Gauss map of* $\mathbf{S}$.

The differential $d\mathbf{N}_p$ of $\mathbf{N}$ at $\mathbf{p} \in \mathbf{S}$ is a linear map from $\mathbf{T}_p(\mathbf{S})$ to $\mathbf{T}_{N(p)}(\mathbf{S}^2)$. Since $\mathbf{T}_p(\mathbf{S})$ and $\mathbf{T}_{N(p)}(\mathbf{S}^2)$ are parallel planes, $d\mathbf{N}_p$ can be looked upon as a linear map on $\mathbf{T}_p(\mathbf{S})$. $d\mathbf{N}_p$ measures how $\mathbf{N}$ pulls away from $\mathbf{N}(\mathbf{q})$ in a neighborhood of $\mathbf{p}$. In the case of curves, this measure is given by a number, the curvature. In the case of surfaces, this measure is characterized by a linear map.

2.8 Second fundamental form II (curvature)

We start with two definitions as follows:

Definition 2.8.1 *The quadratic form II, defined in* $\mathbf{T}_p(\mathbf{S})$ *by* $II_p(v) = - < d\mathbf{N}_p(v), v >$, $v \in \mathbf{T}_p(\mathbf{S})$, *is called the second fundamental form of* $\mathbf{S}$ *at* $\mathbf{p}$.

Definition 2.8.2 *Let* $\mathbf{C}$ *be a regular curve in* $\mathbf{S}$ *passing through* $\mathbf{p} \in \mathbf{S}$, κ *the curvature of* $\mathbf{C}$ *at* $\mathbf{p}$, *and* $\cos\theta =< \mathbf{n}, \mathbf{N} >$, *where* $\mathbf{n}$ *is the normal vector to* $\mathbf{C}$ *and* $\mathbf{N}$ *is the normal vector to* $\mathbf{S}$ *at* $\mathbf{p}$. *The number* $\kappa_n = \kappa \cos\theta$ *is then called the normal curvature of* $\mathbf{C} \subset \mathbf{S}$ *at* $\mathbf{p}$.

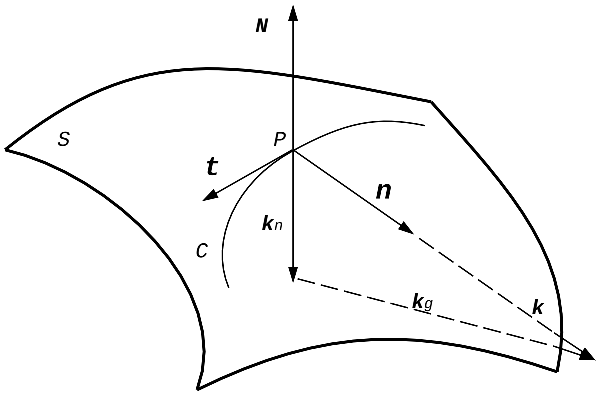


Figure 2.10: Definition of normal curvature

As shown in Figure 2.10, the curvature of a surface S can be quantified in the following way: We consider a curve C on S which passes through point P as shown in Figure 2.10. $\mathbf{t}$ is the unit tangent vector and $\mathbf{n}$ is the unit normal vector of the curve C at point P.

$$\begin{aligned} \frac{d\mathbf{t}}{ds} &= \kappa\mathbf{n} = \mathbf{k}_n + \mathbf{k}_g && (2.13) \\ \mathbf{k}_n &= \kappa_n \mathbf{N} && (2.14) \end{aligned}$$

where $\mathbf{k}_n$ is the normal curvature vector which is normal to the surface, $\mathbf{k}_g$ the geodesic curvature vector tangent to the surface, and $\mathbf{k} = \kappa\mathbf{n}$ the curvature vector of the curve C at point $\mathbf{P}$. κ_n is called the normal curvature of the surface at $\mathbf{P}$ in the direction $\mathbf{t}$.

From Definition 2.8.1, we have

$$\begin{aligned} II &= -d\mathbf{r}\cdot d\mathbf{N} = -(\mathbf{r}_u du + \mathbf{r}_v dv)\cdot(\mathbf{N}_u du + \mathbf{N}_v dv) \\ &= L du^2 + 2M du dv + N dv^2 && (2.15) \end{aligned}$$

where

$$L = \mathbf{N}\cdot\mathbf{r}_{uu}, \quad M = \mathbf{N}\cdot\mathbf{r}_{uv}, \quad N = \mathbf{N}\cdot\mathbf{r}_{vv} \qquad (2.16)$$

The second fundamental form II_p can be interpreted in the following way: Consider a regular curve $\mathbf{C} \subset \mathbf{S}$ parametrized by $\alpha(s)$, where s is the arc length of $\mathbf{C}$, and with $\alpha(0) = \mathbf{p}$. We denote the restriction of the normal vector $\mathbf{N}$ to the curve $\alpha(s)$ by $\mathbf{N}(s)$. Then we have

$$< \mathbf{N}(s), \alpha''(s) >= - < \mathbf{N}'(s), \alpha'(s) > . \qquad (2.17)$$

Therefore,

$$\begin{aligned} II_p(\alpha'(0)) &= - < d\mathbf{N}_p(\alpha'(0)), \alpha'(0) > \\ &= - < \mathbf{N}'(0), \alpha'(0) >=< \mathbf{N}(s), \alpha''(s) > \\ &= < \mathbf{N}, \kappa\mathbf{n} >= \kappa_n(\mathbf{p}) \end{aligned}$$

Namely, the value of the second fundamental form II_p for a unit vector $\mathbf{v} \in \mathbf{T}_p(\mathbf{S})$ is equal to the normal curvature of a regular curve passing through $\mathbf{p}$ and tangent to $\mathbf{v}$.

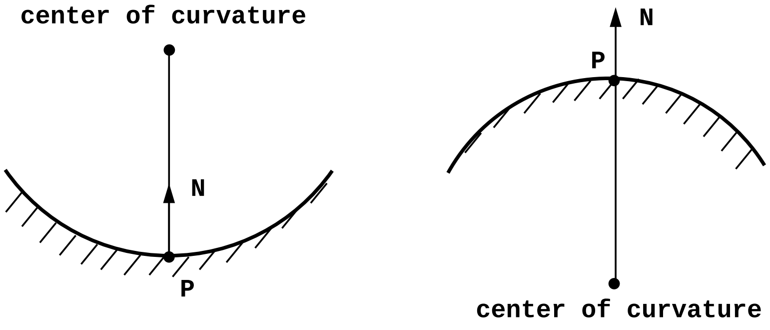
Meusnier's Theorem : All curves lying on a surface S passing through a given point $p \in S$ with the same tangent line have the same normal curvature at this point.

Since $\mathbf{N}\cdot\mathbf{t} = 0$, differentiate w.r.t. s

$$\begin{aligned} \frac{d}{ds}(\mathbf{N}\cdot\mathbf{t}) &= \mathbf{N}'\cdot\mathbf{t} + \mathbf{N}\cdot\mathbf{t}' \\ \frac{d\mathbf{t}}{ds}\cdot\mathbf{N} &= -\mathbf{t}\cdot\frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{N}}{ds} && (2.18) \end{aligned}$$

Recognizing that $ds \cdot ds = dx^2 + dy^2 + dz^2 = d\mathbf{r} \cdot d\mathbf{r}$, we can rewrite Equation 2.18 as:

$$\begin{aligned} \frac{d\mathbf{t}}{ds} \cdot \mathbf{N} &= -\frac{d\mathbf{r} \cdot d\mathbf{N}}{d\mathbf{r} \cdot d\mathbf{r}} \\ \text{while } \frac{d\mathbf{t}}{ds} \cdot \mathbf{N} &= \kappa \mathbf{n} \cdot \mathbf{N} \equiv \kappa_n \end{aligned}$$



(a) **(b)**

Figure 2.11: Definition of positive normal: (a) $\kappa \mathbf{n} \cdot \mathbf{N} = \kappa_n$; (b) $\kappa \mathbf{n} \cdot \mathbf{N} = -\kappa_n$.

Therefore the normal curvature is given by

$$\kappa_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} \tag{2.19}$$

where $\lambda = \frac{dv}{du}$.

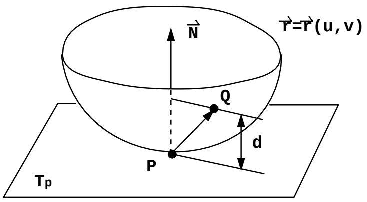


Figure 2.12: Geometrical illustration of the second fundamental form.

Suppose P is a point on a surface and Q is a point in the neighborhood of P, as in Figure 2.12. Taylor's expansion gives

$$\mathbf{r}(u + du, v + dv) = \mathbf{r}(u, v) + \mathbf{r}_u du + \mathbf{r}_v dv + \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + H.O.T. \tag{2.20}$$

Therefore

$$\mathbf{PQ} = \mathbf{r}(u + du, v + dv) - \mathbf{r}(u, v) = \mathbf{r}_u du + \mathbf{r}_v dv + \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + H.O.T.$$

Thus, the projection of $\mathbf{PQ}$ onto $\mathbf{N}$

$$d = \mathbf{PQ} \cdot \mathbf{N} = (\mathbf{r}_u du + \mathbf{r}_v dv) \cdot \mathbf{N} + \frac{1}{2} II$$

and since $\mathbf{r}_u \cdot \mathbf{N} = \mathbf{r}_v \cdot \mathbf{N} = 0$, we get

$$d = \frac{1}{2} II = \frac{1}{2}(L du^2 + 2M du dv + N dv^2)$$

We want to observe in which situation d is positive and negative. When $d = 0$

$$Ldu^2 + 2Mdudv + Ndv^2 = 0$$

Solve for du

$$du = \frac{-M \pm \sqrt{(Mdv)^2 - LNdv^2}}{L} = \frac{-M \pm \sqrt{M^2 - LN}}{L}dv \tag{2.21}$$

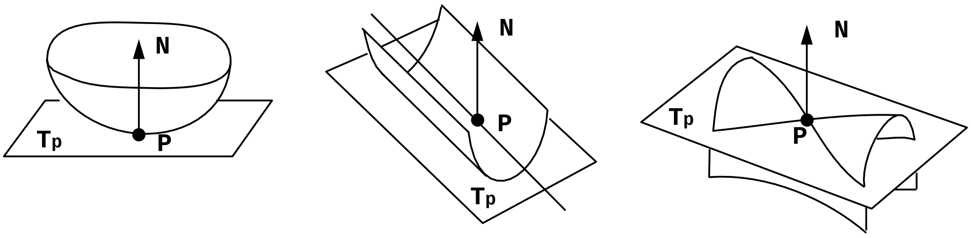


Figure 2.13: (a) Elliptic point; (b) Parabolic point; (c) Hyperbolic point.

- If $M^2 - LN < 0$, there is no real root. That means there is no intersection between the surface and its tangent plane except at point P. P is called *elliptic point* (Figure 2.13(a)).
- If $M^2 - LN = 0$, there is a double root. The surface intersects its tangent plane with one line $du = -\frac{M}{L}dv$, which passes through point P. P is called *parabolic point* (Figure 2.13(b)).
- If $M^2 - LN > 0$, there are two roots. The surface intersects its tangent plane with two lines $du = \frac{-M \pm \sqrt{M^2 - LN}}{L}dv$, which intersect at point P. P is called *hyperbolic point* (Figure 2.13(c)).

2.9 Principal curvatures

For each $\mathbf{p} \in \mathbf{S}$, there exists an orthonormal basis $\{e_1, e_2\}$ of $\mathbf{T}_p(\mathbf{S})$ such that $d\mathbf{N}_p(e_1) = -\kappa_1 e_1$, $d\mathbf{N}_p(e_2) = -\kappa_2 e_2$. Moreover, κ_1 and κ_2 ($\kappa_1 \geq \kappa_2$) are the maximum and minimum of the second fundamental form II_p restircted to the unit circle of $\mathbf{T}_p(\mathbf{S})$. That is, they are the extreme values of the normal curvature at $\mathbf{p}$.

Definition 2.9.1 *The maximum normal curvature κ_1 and the minimum normal curvature κ_2 are called the principal curvatures at $\mathbf{p}$; the corresponding directions, that is, the directions given by the eigenvectors e_1, e_2, are called the principal directions at $\mathbf{p}$.*

From this we have the following definition.

Definition 2.9.2 *If a regular connected curve $\mathbf{C}$ on $\mathbf{S}$ is such that for all $\mathbf{p} \in \mathbf{C}$ the tangent line of $\mathbf{C}$ is a principal direction at $\mathbf{p}$, then $\mathbf{C}$ is said to be a line of curvature of $\mathbf{S}$.*

The extreme values of κ_n can be obtained by evaluating $\frac{d\kappa_n}{d\lambda} = 0$ of Equation 2.19, which gives:

$$(E + 2F\lambda + G\lambda^2)(N\lambda + M) - (L + 2M\lambda + N\lambda^2)(G\lambda + F) = 0 \tag{2.22}$$

Since

$$\begin{aligned} E + 2F\lambda + G\lambda^2 &= (E + F\lambda) + \lambda(F + G\lambda), \\ L + 2M\lambda + N\lambda^2 &= (L + M\lambda) + \lambda(M + N\lambda) \end{aligned}$$

equation (2.22) can be reduced to

$$(E + F\lambda)(M + N\lambda) = (L + M\lambda)(F + G\lambda) \tag{2.23}$$

Thus

$$\kappa_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda} = \frac{L + M\lambda}{E + F\lambda} \tag{2.24}$$

Therefore κ_n satisfies the two simultaneous equations

$$\begin{aligned}(L - \kappa_n E)du + (M - \kappa_n F)dv = 0 \\ (M - \kappa_n F)du + (N - \kappa_n G)dv = 0\end{aligned} \tag{2.25}$$

These equations can be simultaneously satisfied if and only if

$$\begin{vmatrix} L - \kappa_n E & M - \kappa_n F \\ M - \kappa_n F & N - \kappa_n G \end{vmatrix} = 0 \tag{2.26}$$

where $|\ |$ denotes the determinant of a matrix. Expanding and defining K and H as

$$K = \frac{LN - M^2}{EG - F^2} \tag{2.27}$$

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} \tag{2.28}$$

we obtain a quadratic equation for κ_n as follows:

$$\kappa_n^2 - 2H\kappa_n + K = 0 \tag{2.29}$$

The values K and H are called the Gauss (Gaussian) and the mean curvature, respectively.

Definition 2.9.3 *Let* $\mathbf{p} \in \mathbf{S}$ *and let* $d\mathbf{N}_p : \mathbf{T}_p(\mathbf{S}) \to \mathbf{T}_p(\mathbf{S})$ *be the differential of the Gauss map. The determinant of* $d\mathbf{N}_p$ *is the Gaussian curvature* K *of* $\mathbf{S}$ *at* $\mathbf{p}$*. The negative of half of the trace of* $d\mathbf{N}_p$ *is called the mean curvature* H *of* $\mathbf{S}$ *at* $\mathbf{p}$*.*

The discriminant D can be expressed as follows:

$$\begin{aligned} D &= H^2 - K \\ &= \frac{(EN + GL - 2FM)^2 - 4(EG - F^2)(LN - M^2)}{4(EG - F^2)^2} \end{aligned}$$

The denominator is always positive, so we only need to investigate the numerator. The numerator can be written as:

$$\begin{aligned} &(EN + GL - 2FM)^2 - 4(EG - F^2)(LN - M^2) \\ = 4\left(\frac{EG - F^2}{E^2}\right)&(EM - FL)^2 + [EN - GL - \frac{2F}{E}(EM - FL)]^2 \geq 0 \end{aligned}$$

Thus, $D \geq 0$.

Upon solving Equation (2.29) for the extreme values of curvature, we have:

$$\kappa_{max} = H + \sqrt{H^2 - K} \tag{2.30}$$

$$\kappa_{min} = H - \sqrt{H^2 - K} \tag{2.31}$$

From Equations (2.30), (2.31), it is readily seen that

$$K = \kappa_{max}\kappa_{min} \tag{2.32}$$

$$H = \frac{\kappa_{max} + \kappa_{min}}{2} \tag{2.33}$$

From Equation (2.27) (since $EG - F^2 > 0$, see Equation 2.6).

$$\begin{aligned} K > 0 &\Rightarrow LN > M^2 \Rightarrow \text{Elliptic point} \\ K = 0 &\Rightarrow LN = M^2 \Rightarrow \text{Parabolic point} \\ K < 0 &\Rightarrow LN < M^2 \Rightarrow \text{Hyperbolic point} \end{aligned}$$

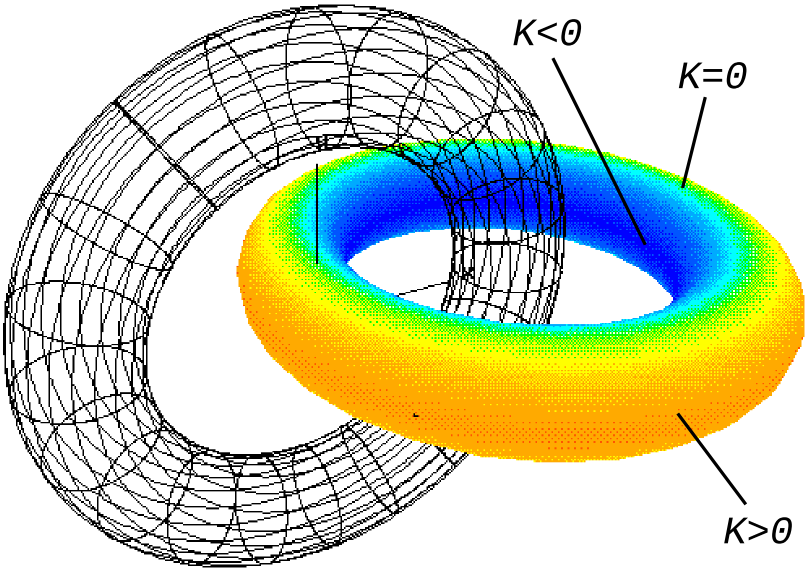


Figure 2.14: Curvature map of a torus showing elliptic, parabolic, and hyperbolic regions.

Definition 2.9.4 *If at* $\mathbf{p} \in \mathbf{S}$, $\kappa_1 = \kappa_2$, *then* $\mathbf{p}$ *is called an umbilical point of* $\mathbf{S}$*; in particular, the planar points are umbilical points.*

From this definition, we can conclude that if all points of a connected surface **S** are umbilical points, then **S** is either contained in a sphere or in a plane.

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